UNSTABLE MANIFOLDS COMPUTATION FOR THE TWO-DIMENSIONAL PLANE POISEUILLE FLOW

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In this work we study the connection among different configurations of the flow in the plane Poiseuille problem in dimension 2. The fluid is confined in a channel of plane parallel walls and modeled by the incompressible Navier-Stokes equations, together with L-periodic boundary conditions ($L = 2\pi/\alpha$, being α the parameter wave number). We have considered the fluid driven by a constant flux or a mean pressure gradient, for which we have respective definitions of the Reynolds number, Re_Q and Re_p. The numerical approximation is detailed in Casas[1]. In figure 1 we represent bifurcating curves of periodic and quasi-periodic flows obtained numerically in Casas[1] using (as in the rest of the paper) 8×70 spectral modes in the spatial variables and $\Delta t = 0.02$ as the time step. Each point of those curves represents the amplitude A. It is also marked on the curves the different stability regions, together with several Hopf bifurcations. In figure 1b it is shown a family of quasi-periodic solutions. The computations have been carried out in a Beowulf cluster of PCs.

In the case of unstable periodic flows we consider the ones that have one real unstable eigenvalue and the remaining ones stable, or a couple of complex conjugate unstable eigenvalues and the remaining ones stable. For each of those periodic flows $u^p(x, y, t)$, we have studied its unstable manifold and which new state of the fluid they are connected to. By means of the Jacobian matrix of the linearized system, we can obtain the eigenvector w associated with the unstable eigenvalue. We have followed the temporal evolution of the perturbed flow $u^p + rw$ until an attracting state is reached. The configurations obtained for several values of Re_p, Re_Q and α are presented in table 1. Unstable periodic solutions for $Re_p \leqslant Re_{p1}$ are connected to the laminar flow. The remaining solutions on the upper branch for both Re_p and Re_Q are attracted by a 2-torus. For $Re_p \geqslant Re_{p1}$ the attractor is the periodic solution on the upper branch for $\alpha = 1.02056$, meanwhile for

$\alpha = 1.1$ this is so for $Re_p \leqslant Re_{p2}$. For $Re_p \geqslant Re_{p2}$ and $\alpha = 1.1$ the solution is connected with 2-torus and more complicated sets. We have an analogous situation for Re_Q and $\alpha = 1.1$. We have followed a similar procedure for the case of unstable quasi-periodic flows, whose results are summarized in table 2. In figure 2 we present the evolution of perturbed flows as a projection of the discrete velocity on the plane of 2 selected coordinates when the flow crosses an appropriate Poincaré section.

Table 1. Attractors of the flow to which is connected the unstable manifold of periodic solutions on the upper and lower branch of the amplitude curve. The temporal evolution is presented in figures 2a and 2b for Re marked with '*' in the table.

UPPER BRANCH									
$\alpha = 1.02056$		$\alpha = 1.02056$		$\alpha = 1.1$		$\alpha = 1.1$		$\alpha = 1.1$	
Re_p	attract.	Re_p	attract.	Re_p	attract.	Re_p	attract.	Re_Q	attract.
4638	laminar	8688	2-torus	3803	laminar	9832	2-torus	5264	2-torus
4654	laminar	9067	2-torus	3816	laminar	10388	2-torus	5402	2-torus
4680	laminar	9478	2-torus	3835	laminar	10990	2-torus	5601	2-torus
6952	2-torus	9921	2-torus	7268	2-torus	11638	2-torus	5801	2-torus
7184	2-torus	10398	2-torus	7615	2-torus			6069	2-torus
7438	2-torus	10912	2-torus	7991	2-torus			6321	2-torus
7713	2-torus			8398	2-torus			6589	2-torus
8012	2-torus			8839	2-torus			6682	2-torus
8336	2-torus			9316	2-torus			6776	2-torus

LOWER BRANCH									
$\alpha = 1.02056$		$\alpha = 1.1$		$\alpha = 1.1$		$\alpha = 1.1$		$\alpha = 1.1$	
Re_p	attract.	Re_p	attract.	Re_p	attract.	Re_Q	attract.	Re_Q	attract.
4636	laminar	3802	laminar	9489	2-torus	3658	periodic	7359	2-torus
4649	laminar	3885	periodic	9589	unkn.	3694	periodic	7639	2-torus
4689	laminar	3969	periodic	10513	2-torus	3816	periodic	7995	3-torus
4722	laminar	4172	periodic	11078	laminar	4020	periodic	8389*	3-torus
4766	periodic	4570	periodic	11375	2-torus	4559	periodic	8682*	unkn.
4821	periodic	4872	periodic			4611	periodic	9045	unkn.
4890	periodic	5272	periodic			4814	periodic	9363	unkn.
4975	periodic	5789	periodic			5101	2-torus	9589	unkn.
5079	periodic	6397	periodic			5500	2-torus	9848	unkn.
5205	periodic	6917	periodic			5822	2-torus	10139	unkn.
5361	periodic	7813	2-torus			6049	2-torus	10390	unkn.
5554	periodic	8192	2-torus			6499	2-torus	10746	unkn.
5772	periodic	8642	2-torus			6791	2-torus	11096	unkn.
		9094	2-torus			7097	2-torus	11395	unkn.

Table 2. Attractors of the flow to which is connected the unstable manifold of quasi-periodic solutions for Re_Q and $\alpha = 1.1$. The temporal evolution of the flow until the attracting solution is reached is presented in figures 2c and 2d for Re marked with '*' in the table.

Re_Q	attractor	Re_Q	attractor	Re_Q	attractor
7953	3-torus	8322*	3-torus	9005*	possible 3-torus
7975	3-torus	8486	3-torus	9096	unknown
8043	3-torus	8623	3-torus	9204	unknown
8157	3-torus	8767	3-torus		
8278	3-torus	8894	possible 3-torus		

References

1. P. S. Casas. *Numerical study of Hopf bifurcations in the two-dimensional plane Poiseuille flow*. PhD thesis, Universidad Politécnica de Cataluña, September 2002. http://www-ma1.upc.es/~casas/research.html.

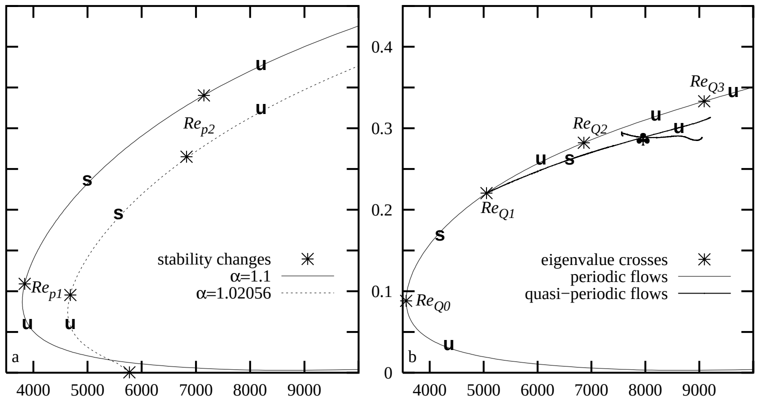

Figure 1. Bifurcating curves of amplitudes for periodic flows based on Re_p (a) and Re_Q (b). The '*' on each curve of (a) corresponds to Hopf bifurcations. They divide the different regions of stability to superharmonic disturbances, which are also labeled in the plot as 's' for "stable" and 'u' for "unstable". The plot in (b) is the analogous of (a) but including a branch of quasi-periodic flows. At Re_{Q0} there is no bifurcation whereas three Hopf bifurcations labeled as Re_{Q1}, Re_{Q2}, and Re_{Q3} are presented on the upper branch. The symbol ♣ indicates a Hopf bifurcation in the curve of quasi-periodic flows.

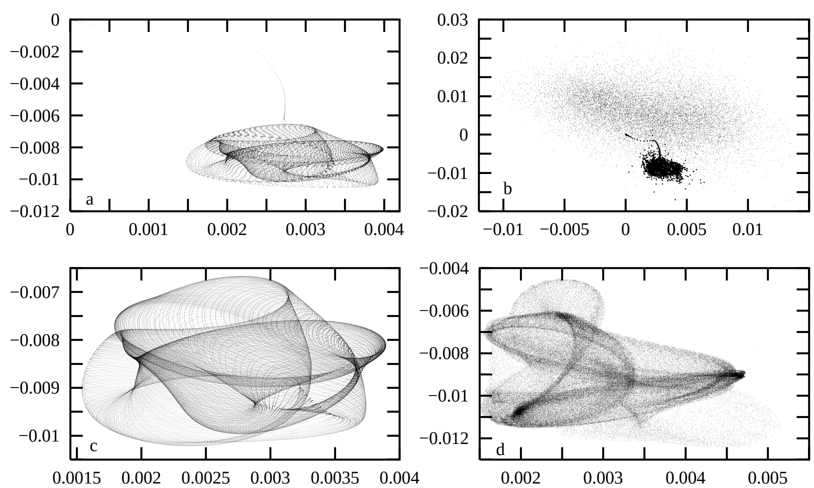

Figure 2. Different time-evolutions of perturbed flows. In (a) and (b) the flow starts from the perturbed unstable periodic solution on the lower branch for $Re_Q = 8389$ and $Re_Q = 8682$ respectively, and $\alpha = 1.1$. In (a) the fluid is first directed to the unstable periodic solution on the upper branch and then attracted by a 3-torus. In (b) the fluid is attracted by a strange set plotted in bigger dots, which is unstable and finally drives the fluid to another strange set. In (c) the perturbed 2-torus for $Re_Q = 8322$ is attracted by a nearly resonant 3-torus, as can be observed. Finally in (d) the perturbed 2-torus for $Re_Q = 9005$ is attracted by a set that reminds a 3-torus.